Computation of External Quality Factors for RF Structures by Means of Model Order Reduction and a Perturbation Approach

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External quality factors are significant quantities to describe losses via waveguide ports in radio frequency resonators. The current contribution presents a novel approach to determine external quality factors by means of a two-step procedure: First, a state-space model for the lossless radio frequency structure is generated and its model order is reduced. Subsequently, a perturbation method is applied on the reduced model so that external losses are accounted for. The advantage of this approach results from the fact that the challenges in dealing with lossy systems are shifted to the reduced order model. This significantly saves computational costs. The present paper provides a short overview on existing methods to compute external quality factors. Then, the novel approach is introduced and validated in terms of accuracy and computational time by means of commercial software.

***Index Terms*—Cavity resonators, numerical simulation, model order reduction.**

I. Introduction

The quality factor is a universal and commonly known measure to quantify losses in resonant systems. It is defined by

$$Q_n = \frac{2\pi f_n W_{\mathrm{sto},n}}{P_{\mathrm{lss},n}}, \tag{1}$$

where f_n is the resonant frequency, $W_{\mathrm{sto},n}$ the energy stored, and $P_{\mathrm{lss},n}$ the averaged loss of energy. All quantities refer to the nth resonance of the structure under consideration. Generally, a variety of loss mechanisms is covered by $P_{\mathrm{lss},n}$.

The largest quality factors occurring in nature are observed at resonances of superconducting cavity resonators [1], such as shown in Fig. 1. These cavity resonators are an essential part of modern particle accelerators. On account of the small surface losses of superconducting structures, the main term contributing to $P_{\mathrm{lss},n}$ are losses of energy via waveguide ports of the structure (red line in Fig. 1). These losses are often referred to as external losses $P_{\mathrm{ext},n}$ and the quality factor accounting for solely external losses is called external quality factor $Q_{\mathrm{ext},n}$. Literature provides a set of different methods to determine external quality factors, which are based on a discrete formulation of Maxwell's equations (see [2] for a rigorous overview). Transient simulations to extract $Q_{\mathrm{ext},n}$ suffer e.g. from long simulation times. Approaches based on scattering parameters require a dense and therefore expensive sampling of scattering spectra because resonances with large quality factors are expressed in narrow-banded peaks in frequency-domain transfer functions. Moreover, a system identification, which is in fact a non-linear optimization, has to be performed. Another way of determining $Q_{\mathrm{ext},n}$ is solving a large non-linear and non-symmetric eigenvalue problem, which arises directly from the discretization. Unfortunately, this scheme is numerically instable and costly. In addition to the mentioned methods, [2] proposes an approach based on eigenmodes of the lossless structure. However, it is reported that the scheme converges slowly depending on the number of considered eigenmodes.

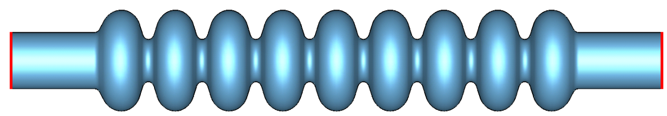

Fig. 1. Superconducting cavity resonator with nine elliptical cells. The boundaries are assumed to be perfect electrically conducting and the inner domain is assumed to be made of vacuum. The red lines mark waveguide ports.

This paper proposes a new approach to compute $Q_{\mathrm{ext},n}$, which overcomes the described drawbacks. First, a state-space model of the lossless structure is generated by means of discretizing the structure. Subsequently, this state-space model is reduced by the employment of a model order reduction approach. The fact that the state-space model describes a lossless structure reduces the computational demand of the model order reduction significantly, i.e. no complex algebra is required and the state-matrix of the state-space system is skew-symmetric. Finally, a perturbation approach is applied to the reduced state-space model to account for external losses. The proposed scheme shifts the hard work of dealing with lossy systems to the reduced model. Hereby, the treatment of losses is significantly simplified.

II. General Theory

In a first step, a large state-space system is delivered by the discretization of the structure under consideration using e.g. the finite integration technique (FIT). This first-order derivative state-space system is reduced in order to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_{\mathrm{rd}}(t) = \mathbf{A}_{\mathrm{rd}}\,\mathbf{x}_{\mathrm{rd}}(t) + \mathbf{B}_{\mathrm{rd}}\,\mathbf{i}(t), \tag{2}$$

$$\mathbf{v}(t) = \mathbf{B}_{\mathrm{rd}}^{\mathrm{T}}\,\mathbf{x}_{\mathrm{rd}}(t). \tag{3}$$

Here, $\mathbf{x}_{\mathrm{rd}}(t)$ is the state vector of the reduced order model, whereas $\mathbf{A}_{\mathrm{rd}}$ and $\mathbf{B}_{\mathrm{rd}}$ are the respective state and input/output matrices. The vectors $\mathbf{v}(t)$ and $\mathbf{i}(t)$ contain modal voltages and currents, which refer to 2D port modes assigned on the cross section of the waveguide ports. The modal voltages of this first-order derivative state-space system are fed back to the modal currents by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_{\mathrm{tc}}(t) = \mathbf{A}_{\mathrm{tc}}\,\mathbf{x}_{\mathrm{tc}}(t) + \mathbf{B}_{\mathrm{tc}}\mathbf{v}(t), \tag{4}$$

$$\mathbf{i}(t) = -\mathbf{C}_{\mathrm{tc}}\,\mathbf{x}_{\mathrm{tc}}(t) - \mathbf{D}_{\mathrm{tc}}\,\mathbf{v}(t). \tag{5}$$

The state-space system (4) - (5) is designed so that the infinite guide termination conditions [3] for all waveguide port modes are accounted for in (2) - (3). In other words, (4) - (5) emulates the short waveguide with constant cross section (refer to Fig. 1) to be infinitely long so that reflections are avoided. Thus, energy can leave the structure and external losses are considered. The matching state-space equations (4) - (5) are interpretable as auxiliary differential equations. Their generation based on Padé approximations with different orders is depicted in detail in [3]. Using (4) - (5) as a feedback for (2) - (3) results in a state-space system, whose non-symmetric state matrix is given by

$$\mathbf{A}_{\mathrm{ma}} = \begin{pmatrix} \mathbf{A}_{\mathrm{rd}} - \mathbf{B}_{\mathrm{rd}}\mathbf{D}_{\mathrm{tc}}\mathbf{B}_{\mathrm{rd}}^{\mathrm{T}} & -\mathbf{B}_{\mathrm{rd}}\mathbf{C}_{\mathrm{tc}} \\ \mathbf{B}_{\mathrm{tc}}\mathbf{B}_{\mathrm{rd}}^{\mathrm{T}} & \mathbf{A}_{\mathrm{tc}} \end{pmatrix} \tag{6}$$

Finally, the frequencies of the loaded resonances and the external quality factors of the structure are determined by

$$f_n = \Im(\underline{\lambda}_n)/2/\pi, \tag{7}$$

$$Q_{\mathrm{ext},n} = \Im(\underline{\lambda}_n)/\Re(\underline{\lambda}_n)/2, \tag{8}$$

whereas $\underline{\lambda}_n$ are the complex eigenvalues of $\mathbf{A}_{\mathrm{ma}}$ and $\Im(\underline{\lambda}_n)$ and $\Re(\underline{\lambda}_n)$ its imaginary and real part, respectively. Despite the fact that $\mathbf{A}_{\mathrm{ma}}$ is non-symmetric, the computation of its eigenvalues is straightforward due to the size of the matrix.

III. Numerical Validation

To validate the proposed approach, the cavity resonator depicted in Fig. 1 is discretized with a hexahedral mesh by means of FIT [4], using three symmetry planes. The symmetry conditions are chosen so that TM monopole and TE quadrupole modes are considered. This results in two (one for each symmetry condition on the transverse symmetry plane of the cavity) state-space models with $651{,}726$ degrees of freedom. They are reduced to state-space models with 70 degrees of freedom. For the actual validation, (4) - (5) are constructed based on a zeroth order Padé approximation (simplest case). This special case is identical to the approach proposed in [5]. It delivers two matrices $\mathbf{A}_{\mathrm{ma}}$ (one for each symmetry condition) with 70 rows and columns. Thus, the computation of the eigenvalues and the final determination of f_n and $Q_{\mathrm{ext},n}$ is computationally inexpensive. Fig. 2 shows the external quality factors of resonances belonging to the TM monopole and to the TE quadrupole bands. Red crosses denote the external quality factors computed by the proposed approach (total computational time: $10\,\mathrm{min}$), whereas the blue dots show the external quality factors directly computed with the JDM eigenmode solver of [4] (total computational time: $1\,\mathrm{h}\ 15\,\mathrm{min}$). According to the documentation, [4] determines the external quality factors from the lossless eigenmodes of the structure in a post-processing step. Unfortunately, the underlying method is not described in the documentation of the program. All computational times mentioned here are wall-clock times and refer to a computer equipped with an Intel(R) Core(TM) i5-2400 CPU @ $3.10\,\mathrm{GHz}$ and $8\,\mathrm{GB}$ RAM.

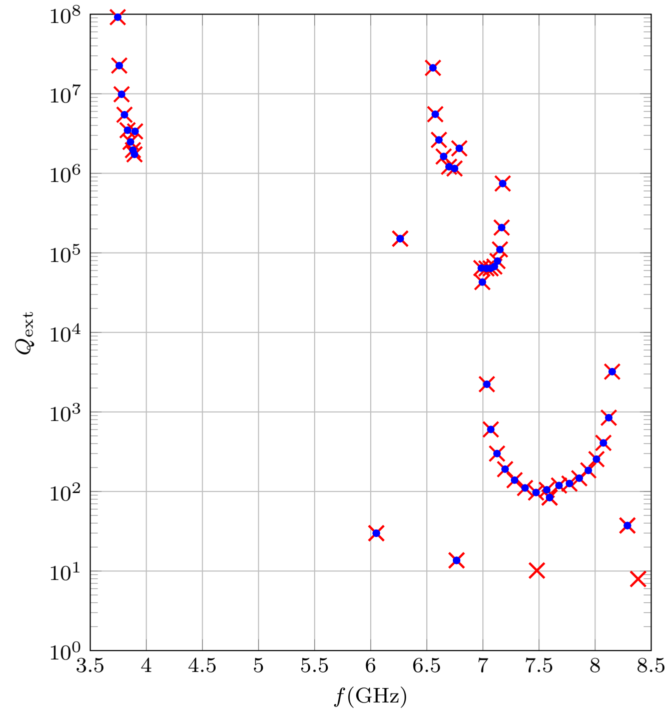

Fig. 2. External quality factors of monopole and quadrupole modes of the cavity depicted in Fig. 1. The blue dots denote the external quality factors delivered by the eigenmode solver of [4]. The red crosses represent the quality factors obtained by the proposed approach. For the comparison a 0th order Padé approximation is used to model the infinite guide termination condition, i.e. (5) is simplified to $\mathbf{i}(t) = -\mathbf{D}_{\mathrm{tc}}\,\mathbf{v}(t)$.

IV. Summary and Outlook

The results delivered by the proposed method coincide well with results obtained by commercial software while saving computational time. The full contribution describes the entire approach in a more detailed manner.

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